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Wake-Like Solutions of the Laminar Boundary-Layer Equations

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A class of wake-like solutions of the Falkner-Skan equation $f''' + ff'' - \beta(f'^2 - 1) = 0$ satisfying the boundaı y conditions $f(0) = f''(0) = 0, f'(\infty) = 1$ has been uncoveı ed and the solutions tabulated for 11 values of the pressure-gradient parameter β For $-0\ 1988 < \beta < 0$, the velocity at the origin [i e , $f'(0)$] is negative and reaches a minimum value of $-0\ 1843$ at $\beta = -0\ 0352$ At $\beta = -0\ 1988$ the new solution coincides with the separation solution calculated by Hartree, the difference being that the new solution furnishes $f'(0) = 0$ as a result The new solution for $\beta \to 0$ is shown to approach Chapman's solution (with a shift of origin) for the constant pressure flow past an infinite step The nonuniqueness problem encountered by Hartree for $\beta < 0$ did not appear in the calculation of the new solution

IN 1950, Chapman[1] found a solution of the two-dimen sional laminar boundary-layer equation which is applicable to the constant-pressure flow past a step (Fig 1) In a subsequent paper[2] he showed that this solution satisfied the Blasius equation, $f''' + ff'' = 0$ and the boundary conditions $f(0) = 0, f'(\infty) = 1, f'(-\infty) = 0$ Now the Blasius equation with the boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$ governs the constant-pressure flow along a sur face, and its generalization, the Falkner-Skan equation[3]

$$f''' + ff'' - \beta(f'^2 - 1) = 0 \tag{1}$$

allows a certain class of streamwise pressure gradients Hartree[4] has calculated and tabulated solutions of the Falkner-Skan equation for a range of values of the pressure-gradient parameter β Although originally derived for in compressible flow, the Falkner-Skan equation has been generalized by Illingworth[5] and Stewartson[6] to include compressible flow Thus, Cohen and Reshotko,[7] following the foregoing methods, give the set

$$f''' + ff'' - \beta(f'^2 - 1 - S) = 0 \tag{2a}$$

$$S'' + fS' = 0 \tag{2b}$$

for the boundary-layer flow of a compressible fluid with Prandtl number unity The variable S is the stagnation-enthalpy parameter $(H - H)/H$ In the light of what has been said one may ask whether there are other solutions of the Falkner-Skan equation bearing to Chapman's constant-pressure solution the same relationship as Hartree's solu tions bear to the constant pressure Blasius solution

The flow immediately downstream of a flat-based body of finite dimension normal to the stream is as sketched in Fig 2a The streamline MN separates the flow that circulates in a vortex motion near the base from the flow which passes on downstream The shape of this boundary is determined by the interaction of the inner flow, which is fundamentally viscous, with the flow far from the axis, which may be considered inviscid The point N is a free stagnation point at which the shear and pressure forces just balance Since the point P is also a stagnation point there is a point on PN where $\partial u/\partial x = 0$, and hence again the shear and pressure forces balance Thus the product $u\ (\partial u/\partial x)$ is at first positive on PQ, then negative on QN, zero at N, and positive thereafter, as shown in Fig 2b Since the viscous term is always positive, the vicinity of P is a region where viscous effects predominate, the pressure forces coming into play as Q is approached, becoming predominant in QN, and finally stagnating the flow at N Downstream of N the external flow is very nearly parallel to the axis so there can be no strong pressure gradients and the viscous effects again pre dominate In view of the alternating dominance of the viscous and pressure terms it is unlikely that any analysis neglecting either one will yield good results

Assuming for the moment the applicability of Eq (2) to the flow model just discussed, it remains to establish suitable boundary conditions at the origin, which obviously is the axis of symmetry PN in Fig 2a These conditions must enforce symmetry about the axis, i e , $f(0) = f''(0) = 0$, and $S'(0) = 0$ The outer boundary conditions are $f'(\infty) = 1$ and $S(\infty) = 0$ The obvious solution of Eq (2b) satisfying these

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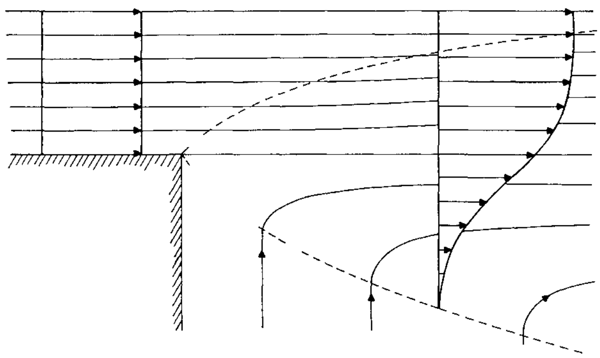

Fig 1 Chapman's constant pressure model

boundary conditions is $S \equiv 0$ everywhere and, therefore, we need consider only Eq (1) with the boundary conditions

$$f(0) = f''(0) = 0 \qquad f'(\infty) = 1 \tag{3}$$

For any given value of β the problem is to find a value of $f'(0)$ that is compatible with the outer boundary condition In this our problem differs from that of Hartree, in that in his solutions $f''(0)$ is the parameter and $f'(0)$ is known to vanish However, for the particular value $\beta = -0\ 1988$, Hartree found a solution giving $f''(0) = 0$, which of course satisfies our boundary conditions Therefore, in our solutions for this value of β, the result $f'(0) = 0$ must be expected

The approach of the new solutions to Chapman's solution as $\beta \to 0$ is not so obvious and ultimately will be established on the basis of the numerical results However, certain similarities present themselves and indeed were responsible for originating the investigation Thus, referring to Fig 2a, it is evident that with the value $f = 0$ on the axis, by definition the value $f = 0$ is reached again on the bounding streamline MN, i e , the net mass flux of the inner flow vanishes If Chapman's $f(0) = 0$ is taken as this outer zero then his origin is translated from ours and his third boundary condition $f'(-\infty) = 0$ must be reconciled with our $f''(0) = 0$ These, of course, are not, on their face, incompatible if it can be established that for the new solutions the distance between the two points where $f = 0$ increases without limit, and furthermore, that $f'(0) \to 0$ as $\beta \to 0$ In the absence of an analytical solution these results can only be established by numerical integration of Eq (1) for various values of β Following Hartree's original scheme, the idea is to pick various values of $f'(0)$ for a particular β and integrate out from the origin until f' is sufficiently close to unity However, it is obvious that for $\beta \equiv 0$ any such scheme must fail since the second- and all higher-order derivatives vanish at the origin in that case Thus, there is further evidence of the correspondence of Chapman's solution with a special case of ours

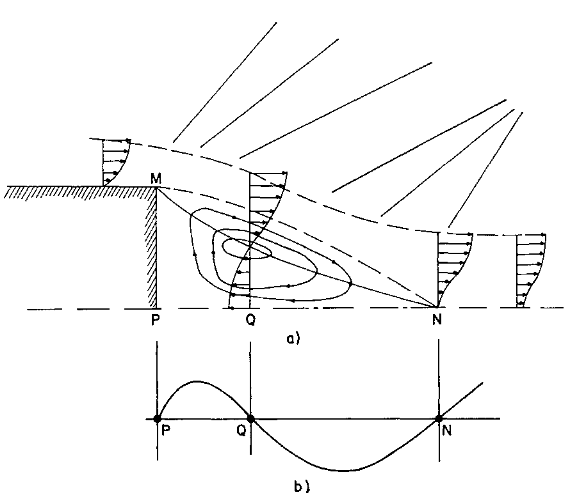

Fig 2 Reversed flow model behind finite body

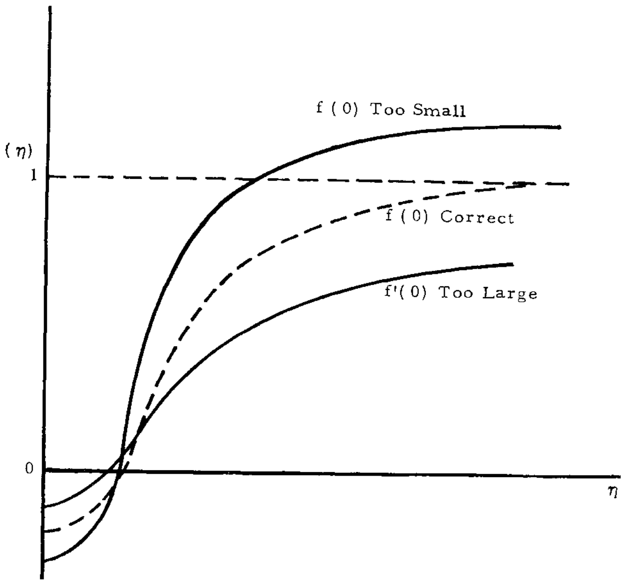

Fig 3 Sketch of typical trial curves

(with a shift of origin), since Chapman's f' is asymptotic to zero (i e , all higher order derivatives vanish at $-\infty$ in his problem)

In addition to the correspondence with Chapman's solution and Hartree's solution for $\beta = -0\ 1988$, there is the solution $f'(\eta) \equiv 1$ which satisfies the differential equation (1) and the boundary conditions (3) The value of β corresponding to this solution cannot be found by any numerical procedure but is shown in the Appendix to be $-0\ 5$ Thus, three points are known in the $f'(0) \sim \beta$ plane ($\beta = 0, -0\ 1988$ and $-0\ 5$) and at the first two $f'(0) = 0$ The stationary point in $0 \geq \beta \geq -0\ 1988$ is obviously a minimum (since $f'(0) = 1$ for $\beta = -0\ 5$) and in this range $f'(0)$ is negative This is not entirely unexpected since the physical model requires negative velocities along PN in Fig 2a The solution with $\beta = -0\ 1988$ is appropriate at the stagnation point Since $f'''(0) = -\beta(1 - f'(0)^2)$ and $f'(\eta) = f'(0) + f'''(0)\eta^2$ near the origin, it follows that whether $f'(\eta)$ is greater or less than unity, β must be negative if $f'(\eta)$ is to vary monotonically from $f'(0)$ at the origin to unity at the outer edge

Computed Results

The solution of Eq (2b) with the homogeneous boundary conditions $S'(0) = 0$, $S(\infty) = 0$ yields the solution $S(\eta) \equiv 0$ everywhere, and so the problem reduces to solving the Falkner-Skan equation, Eq (1), with the boundary conditions $f(0) = f''(0) = 0$, $f'(\infty) = 1$ The complexity of the equation necessitates a numerical solution The calculation procedure was to guess a value of $f'(0)$ using the rough form

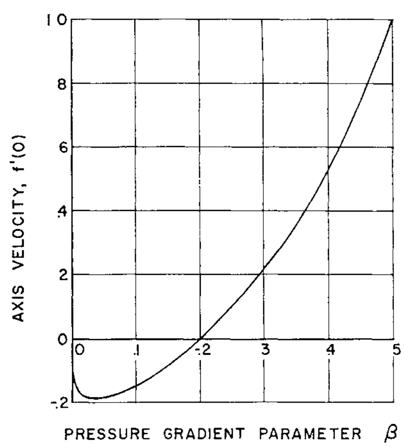

Fig 4 Relationship between $f'(0)$ and β

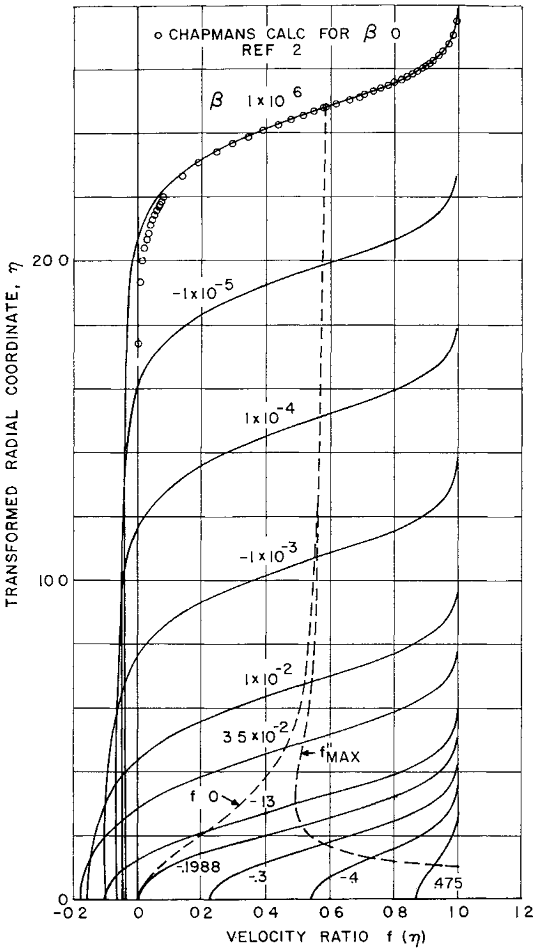

Fig 5 Velocity profiles for various values of β

of the $f'(0) \sim \beta$ curve as a guide and then to integrate in the direction of increasing η until either $f'(\eta)$ became larger than unity or $f''(\eta)$ became less than a small number (This number was arbitrarily chosen at 10^{-5}) If the first criterion was satisfied but not the second, the value of $f'(0)$ was in creased; the opposite occurred if the second, but not the first, was satisfied The solution was determined when $1 - f'$ and f'' were simultaneously less than 10^{-5} The first runs took as many as twenty integrations, but as the $f'(0) \sim \beta$ curve was developed the initial guesses became more accurate and the calculation time decreased As is characteristic of boundary-layer problems, the approach of f' to its asymptotic value is quite rapid and the significant range of the independent variable η is finite Typical trial curves are shown in Fig 3 Once the correct value of $f'(0)$ was bracketed by two trials, the interval in $f'(0)$ was halved until the correct solution was obtained The program for the integration was written for the IBM 7090 computer and provided for three options in the method of integration Thus, there were available routines for the Runge-Kutta method with fixed step size and the predictor-corrector method with either fixed or variable step size All three options were used at various times

As a result of the numerical integrations there has emerged a distinction of fundamental importance between the new solutions and those of Hartree Hartree found that when β was negative the asymptotic behavior of the solution was not uniquely related to the parameter $f''(0)$ of his problem, and indeed there was a range of values of $f''(0)$ for any negative β that would yield $f'(\infty) = 1$ Of this range, Hartree chose that one which gave the most rapid approach of $f'(\eta)$ to unity This solution was tabulated in Ref 4 and was recalculated by Smith in Ref 8 This nonuniqueness did not occur with the new solutions Any value of $f'(0)$ smaller than the correct one gave values of $f'(\eta)$ greater than unity, and a value larger than the correct one gave a corresponding asymptotic value of f' less than unity

Figure 4 shows the variation of $f'(0)$ with the pressure-gradient parameter β The minimum value of $-0\ 1843$ is reached at $\beta = -0\ 0352$ The range of $0 \geq \beta \geq -0\ 1988$ is of particular interest because of the possibility of reversed flow It is seen that any small negative β yields an appreciable velocity on the axis (At $\beta = -1\ 0 \times 10^{-6}$, $f'(0)$ is $-0\ 0396$) The velocity profiles for selected values of β are shown in Fig 5 along with the locus of the outer points $f = 0$ and the locus of points of maximum shear (i e , f'' max) As $\beta \rightarrow 0$, these two loci are asymptotic to each other and to the value $f' = 0\ 587$, which Chapman found to be the value at the origin of his solution The locus of points $f = 0$ starts at $\eta = 0$ for $\beta = -0\ 1988$ while that of f'' max begins at $\eta = 1$ for $\beta = -0\ 5$ (This limiting value can be calculated from the Appendix) Interestingly enough, the approach of the locus of the maximum shear points to the asymptotic value of $f'(\eta) = 0\ 587$ is not monotonic, but a minimum velocity ratio (i e , f') of 0 487 is reached at $\beta = -0\ 128$ Of course for $\beta < -0\ 1988$ the velocity on the axis is positive and the outer point $f = 0$ no longer exists

As further evidence of the asymptotic approach of our solutions to Chapman's solution, his solution is superimposed on the new one (for $\beta = -1 \times 10^{-6}$) in Fig 5 For this value of β, the numerical solution gave $f = 0$ at $\eta = 24\ 80$ and according to what has been said earlier this was taken as the origin of Chapman's solution The circled points on Fig 5 are from Table 1 of Ref 2 with the independent variable multiplied by $2^{1/2}$ in order to agree with our definition The solutions are seen to coincide over most of the range of velocity; however, differences appear as the axis ($\eta = 0$) is approached This is because even for this extremely small value of β the velocity on the axis is some 20% of its peak negative value and is not insignificant However, there is no doubt that the Chapman solution is a special case of ours and is being approached as $\beta \rightarrow 0$

The calculated values of f, f', and f'' (to four figures) are given in Tables 1–11 for the values of β shown in Fig 5

Table 1 $\beta = -1\ 0 \times 10^{-6}$

η	f	f'	f''
0	0	−0 0396	0
1 0	−0 0396	−0 0396	0
2 0	−0 0792	−0 0396	0
3 0	−0 1187	−0 0396	0
4 0	−0 1584	−0 0396	0
5 0	−0 1980	−0 0396	0
6 0	−0 2376	−0 0396	0
7 0	−0 2771	−0 0396	0
8 0	−0 3167	−0 0395	0
9 0	−0 3962	−0 0395	0
10 0	−0 3957	−0 0395	0
11 0	−0 4352	−0 0394	0
12 0	−0 4746	−0 0393	0 0001
13 0	−0 5139	−0 0392	0 0002
14 0	−0 5530	−0 0390	0 0003
15 0	−0 5918	−0 0386	0 0005
16 0	−0 6300	−0 0378	0 0010
17 0	−0 6672	−0 0364	0 0019
18 0	−0 7024	−0 0337	0 0038
19 0	−0 7338	−0 0282	0 0077
20 0	−0 7570	−0 0168	0 0163
21 0	−0 7631	+0 0076	0 0349
22 0	−0 7326	+0 0598	0 0741
23 0	−0 6251	+0 1673	0 1473
24 0	−0 3678	+0 3640	0 2461
25 0	+0 1307	+0 6396	0 2835
26 0	+0 8991	+0 8781	0 1729
27 0	+1 8390	+0 9801	0 0443
28 0	+2 8317	+0 9986	0 0043
29 0	+3 8313	+1 0000	0 0001

Table 2 $\beta = -1\ 0 \times 10^{-5}$

η	f	f'	f''
0	0	−0 0513	0
1 0	−0 0513	−0 0513	0
2 0	−0 1025	−0 0512	0
3 0	−0 1537	−0 0512	0
4 0	−0 2049	−0 0512	0
5 0	−0 2561	−0 0511	0 0001
6 0	−0 3071	−0 0510	0 0001
7 0	−0 3581	−0 0509	0 0002
8 0	−0 4088	−0 0506	0 0003
9 0	−0 4593	−0 0503	0 0004
10 0	−0 5094	−0 0498	0 0007
11 0	−0 5588	−0 0488	0 0012
12 0	−0 6069	−0 0472	0 0022
13 0	−0 6528	−0 0442	0 0041
14 0	−0 6944	−0 0384	0 0080
15 0	−0 7277	−0 0268	0 0163
16 0	−0 7439	−0 0026	0 0341
17 0	−0 7242	+0 0480	0 0715
18 0	−0 6302	+0 1516	0 1420
19 0	−0 3916	+0 3424	0 2406
20 0	+0 0836	+0 6162	0 2871
21 0	+0 8323	+0 8638	0 1855
22 0	+1 7638	+0 9763	0 0511
23 0	+2 7548	+0 9982	0 0093
24 0	+3 7543	+0 9999	0 0002
29 0	+4 7543	+1 0000	0

Table 3 $\beta = -1\ 0 \times 10^{-4}$

η	f	f'	f''
0	0	−0 0713	0
1 0	−0 0713	−0 0712	0 0001
2 0	−0 1425	−0 0711	0 0002
3 0	−0 2134	−0 0708	0 0004
4 0	−0 2840	−0 0703	0 0006
5 0	−0 3540	−0 0696	0 0009
6 0	−0 4230	−0 0684	0 0015
7 0	−0 4905	−0 0664	0 0025
8 0	−0 5555	−0 0631	0 0043
9 0	−0 6160	−0 0572	0 0079
10 0	−0 6682	−0 0461	0 0152
11 0	−0 7047	−0 0243	0 0304
12 0	−0 7095	+0 0200	0 0619
13 0	−0 6498	+0 1096	0 1233
14 0	−0 4638	+0 2784	0 2183
15 0	−0 0609	+0 5396	0 2901
16 0	+0 6197	+0 8105	0 2245
17 0	+1 5174	+0 9599	0 0781
18 0	+2 5010	+0 9963	0 0105
19 0	+3 4998	+0 9999	0 0005

It is not without interest to compare our results for $\beta = -0\ 19883$ with those of Smith,[8] who was of course calculating a different problem, which in this special case coincides with ours The two calculations agree to the number of significant figures shown, but the methods of computation were quite different It is of particular interest that whereas Smith was faced with the nonuniqueness problem discovered by Hartree, in our calculation the value $f'(0) = -1\ 953 \times 10^{-6}$ was calculated as the unique solution This of course corresponds to the $f'(0) \equiv 0$ of Smith's solution

The shear-velocity diagram is shown in Fig 6 along with Chapman's solution (obtained by three point numerical differentiation of the data in Table 1 of Ref 2) As noted previously, the shear stress [as characterized by $f''(\eta)$] reaches a maximum in $0 < \eta < \infty$ The locus of these maxima also has a peak as is shown in Fig 6 The maximum shear stress increases from Chapman's value (0 282) to a maximum of 0 362 at $\beta = -0\ 177$ and thereafter falls to zero at $\beta = -0\ 5$ The velocity ratio $f'(\eta)$ at the maximum value is 0 495 The maximum shear and the velocity at which it occurs is plotted against the pressure gradient parameter β in Fig 7

Table 4 $\beta = -1\ 0 \times 10^{-3}$

η	f	f'	f'
0	0	−0 1074	0
0 5	−0 0537	−0 1073	0 0005
1 0	−0 1072	−0 1069	0 0010
1 5	−0 1605	−0 1062	0 0016
2 0	−0 2134	−0 1052	0 0023
2 5	−0 2657	−0 1039	0 0031
3 0	−0 3172	−0 1021	0 0041
3 5	−0 3677	−0 0997	0 0054
4 0	−0 4168	−0 0966	0 0072
4 5	−0 4641	−0 0925	0 0095
5 0	−0 5091	−0 0870	0 0127
5 5	−0 5508	−0 0796	0 0171
6 0	−0 5883	−0 0696	0 0233
6 5	−0 6198	−0 0559	0 0321
7 0	−0 6433	−0 0369	0 0446
7 5	−0 6555	−0 0105	0 0623
8 0	−0 6520	+0 0265	0 0871
8 5	−0 6266	+0 0780	0 1206
9 0	−0 5708	+0 1487	0 1634
9 5	−0 4740	+0 2427	0 2132
10 0	−0 3239	+0 3617	0 2614
10 5	−0 1088	+0 5013	0 2926
11 0	+0 1787	+0 6484	0 2887
11 5	+0 5374	+0 7828	0 2423
12 0	+0 9961	+0 8858	0 1673
12 5	+1 4167	+0 9501	0 0926
13 0	+1 9009	+0 9823	0 0404
13 5	+2 3957	+0 9949	0 0138
14 0	+2 8944	+0 9988	0 0037
14 5	+3 3941	+0 9998	0 0008

Table 5 $\beta = -1\ 0 \times 10^{-2}$

η	f	f'	f''
0	0	−0 1637	0
0 9	−0 0816	−0 1625	0 0049
1 0	−0 1620	−0 1587	0 0103
1 5	−0 2398	−0 1520	0 0165
2 0	−0 3135	−0 1419	0 0242
2 5	−0 3810	−0 1274	0 0342
3 0	−0 4400	−0 1071	0 0475
3 5	−0 4869	−0 0791	0 0655
4 0	−0 5173	−0 0405	0 0900
4 5	−0 5251	+0 0102	0 1226
5 0	−0 5020	+0 0836	0 1644
5 5	−0 4376	+0 1779	0 2139
6 0	−0 3200	+0 2977	0 2643
6 5	−0 1361	+0 4401	0 3017
7 0	+0 1223	+0 5940	0 3073
7 5	+0 4566	+0 7400	0 2692
8 0	+0 8575	+0 8573	0 1958
8 5	+1 3071	+0 9346	0 1148
9 0	+1 6887	+0 9696	0 0634
9 5	+2 2787	+0 9926	0 0194
10 0	+2 7766	+0 9982	0 0055
10 5	+3 2762	+0 9996	0 0012

The variations of the momentum thickness θ and the displacement thickness δ^* with β are shown in Figs 8 and 9, respectively The momentum thickness vanishes for $\beta = 0$ and $-0\ 5$ and has a peak value of 0 631 at $\beta = -0\ 275$ The displacement thickness increases monotonically from zero as β increases from $-0\ 5$

Figures 8 and 9 also show Smith's values for the ordinary boundary-layer solutions in $-0\ 1988 < \beta < 0$ In this range the wake solutions have a smaller momentum thickness (by virtue of the region of reversed flow) and a larger displacement thickness

Conclusions

A new set of solutions of the Falkner-Skan equation satisfying the boundary conditions $f(0) = f''(0) = 0\ f'(\infty) = 1$

Table 6 $\beta = -0\ 035$

η	f	f'	f''
0	0	−0 1843	0
0 5	−0 0915	−0 1801	0 0172
1 0	−0 1786	−0 1669	0 0360
1 5	−0 2967	−0 1435	0 0583
2 0	−0 3201	−0 1077	0 0859
2 5	−0 3618	−0 0564	0 1210
3 0	−0 3731	+0 0147	0 1648
3 5	−0 3431	+0 1098	0 2166
4 0	−0 2588	+0 2317	0 2707
4 5	−0 1071	+0 3788	0 3141
5 0	+0 1225	+0 5408	0 3276
5 5	+0 4330	+0 6987	0 2959
6 0	+0 8167	+0 8297	0 2231
6 5	+1 2558	+0 9194	0 1361
7 0	+1 7293	+0 9687	0 0658
7 5	+2 2199	+0 9902	0 0249
8 0	+2 7172	+0 9975	0 0073
8 5	+3 2166	+0 9995	0 0017
9 0	+3 7166	+0 9999	0 0003
9 5	+4 2164	+1 0000	0

Table 7 $\beta = -0\ 13$

η	f	f'	f''
0	0	−0 1066	0
0 2	−0 0211	−0 1040	0 0257
0 4	−0 0413	−0 0963	0 0917
0 6	−0 0593	−0 0833	0 0782
0 8	−0 0742	−0 0690	0 1053
1 0	−0 0849	−0 0411	0 1331
1 2	−0 0903	−0 0117	0 1617
1 4	−0 0892	+0 0236	0 1909
1 6	−0 0805	+0 0647	0 2203
1 8	−0 0629	+0 1117	0 2495
2 0	−0 0394	+0 1644	0 2776
2 2	+0 0032	+0 2226	0 3036
2 4	+0 0539	+0 2857	0 3263
2 6	+0 1177	+0 3528	0 3438
2 8	+0 1952	+0 4228	0 3550
3 0	+0 2869	+0 4943	0 3584
3 2	+0 3929	+0 5656	0 3530
3 4	+0 5130	+0 6348	0 3383
3 6	+0 6466	+0 7003	0 3149
3 8	+0 7928	+0 7603	0 2839
4 0	+0 9503	+0 8135	0 2476
4 2	+1 1177	+0 8992	0 2083
4 4	+1 2934	+0 8968	0 1689
4 6	+1 4759	+0 9268	0 1318
4 8	+1 6637	+0 9498	0 0989
5 0	+1 8554	+0 9668	0 0713
5 2	+2 0501	+0 9788	0 0494
5 4	+2 2467	+0 9869	0 0328
5 6	+2 4446	+0 9922	0 0209
5 8	+2 6434	+0 9955	0 0128
6 0	+2 8427	+0 9975	0 0075
6 2	+3 0424	+0 9987	0 0043
6 4	+3 2422	+0 9993	0 0023
6 6	+3 4421	+0 9997	0 0012
6 8	+3 6421	+0 9998	0 0006
7 0	+3 8420	+0 9999	0 0003

has been calculated Although these solutions are strictly applicable only where the pressure varies as a power of the streamwise distance, it may be possible to use them to develop approximate methods of wider applicability as has been done in the analogous case of the Hartree solutions

Although the Falkner-Skan equation was originally derived for incompressible flow, the Illingworth and Stewartson transformations have extended its range of applicability to compressible flows of a perfect gas and it can be shown that, if the product $\rho\mu$ is constant, it applies also to a real gas in thermodynamic equilibrium Thus, the new solutions apply to a wide range of two-dimensional problems

Table 8 $\beta = -0\ 1988$

η	f	f'	f''
0	0	0	0
0 2	0 0003	0 0040	0 0398
0 4	0 0021	0 0159	0 0795
0 6	0 0072	0 0358	0 1192
0 8	0 0170	0 0635	0 1985
1 0	0 0331	0 0991	0.1971
1 2	0 0571	0 1423	0 2344
1 4	0 0905	0 1927	0 2693
1 6	0 1346	0 2498	0 3008
1 8	0 1908	0 3127	0 3272
2 0	0 2600	0 3802	0 3470
2 2	0 3431	0 4509	0 3586
2 4	0 4405	0 5231	0 3608
2 6	0 5523	0 5946	0 3527
2 8	0 6782	0 6634	0 3344
3 0	0 8174	0 7277	0 3070
3 2	0 9688	0 7858	0 2723
3 4	1 1311	0 8363	0 2329
3 6	1 3028	0 8788	0 1918
3 8	1 4822	0 1913	0 1519
4 0	1 6676	0 9398	0 1156
4 2	1 8976	0 9597	0 0845
4 4	2 1488	0 9794	0 0488
4 6	2 2469	0 9838	0 0398
4 8	2 4444	0 9903	0 0257
5 0	2 6429	0 9944	0 0159
5 2	2 8420	0 9969	0 0095
5 4	3 0416	0 9983	0 0054
5 6	3 2413	0 9991	0 0029
5 8	3 4412	0 9996	0 0015
6 0	3 6412	0 9998	0 0008
6 2	3 8411	0 9999	0 0004
6 4	4 0411	0 9999	0 0002
6 6	4 2411	1 0000	0
6 8	4 4411	1 0000	0
7 0	4 6411	1 0000	0

Table 9 $\beta = -0\ 30$

η	f	f'	f''
0	0	0 2211	0
0 2	0 0446	0 2268	0 0568
0 4	0 0915	0 2438	0 1124
0 6	0 1428	0 2716	0 1651
0 8	0 2008	0 3095	0 2135
1 0	0 2672	0 3566	0 2558
1 2	0 3439	0 4113	0 2903
1 4	0 4322	0 4720	0 3151
1 6	0 9330	0 5366	0 3287
1 8	0 6469	0 6027	0 3303
2 0	0 7740	0 6679	0 3198
2 2	0 9139	0 7299	0 2982
2 4	1 0655	0 7866	0 2677
2 6	1 2281	0 8365	0 2309
2 8	1 3997	0 8288	0 1914
3 0	1 5790	0 9131	0 1521
3 2	1 7645	0 9399	0 1160
3 4	1 9545	0 9598	0 0847
3 6	2 1480	0 9741	0 0593
3 8	2 3439	0 9840	0 0398
4 0	2 5414	0 9904	0 0256
4 2	2 7399	0 9945	0 0157
4 4	2 9391	0 9969	0 0092
4 6	3 1386	0 9984	0 0054
4 8	3 3384	0 9991	0 0029
5 0	3 5388	0 9996	0 0015
5 2	3 7382	0 9998	0 0007
5 4	3 9382	0 9999	0 0004
5 6	4 1382	0 9999	0 0002
5 8	4 3381	1 0000	0

As a possible direction for extension of these solutions, there may be indicated flows with Prandtl number other than unity and axially symmetric flows Of these the second is

Table 10 $\beta = -0\ 40$

η	f	f'	f''
0	0	0 5367	0
0 2	0 1077	0 5423	0 0564
0 4	0 2177	0 5590	0 1094
0 6	0 3320	0 5857	0 1559
0 8	0 4525	0 6207	0 1930
1 0	0 5807	0 6621	0 2187
1 2	0 7176	0 7074	0 2318
1 4	0 8637	0 7540	0 2323
1 6	1 0191	0 7995	0 2212
1 8	1 1833	0 8418	0 2007
2 0	1 3555	0 8794	0 1738
2 2	1 5347	0 9112	0 1438
2 4	1 7196	0 9369	0 1137
2 6	1 9091	0 9568	0 0859
2 8	2 1020	0 9715	0 0620
3 0	2 2974	0 9819	0 0428
3 2	2 4945	0 9889	0 0283
3 4	2 6928	0 9935	0 0179
3 6	2 8918	0 9963	0 0108
3 8	3 0912	0 9980	0 0063
4 0	3 2909	0 9989	0 0035
4 2	3 4908	0 9995	0 0018
4 4	3 6907	0 9997	0 0009
4 6	3 8907	0 9999	0 0005
4 8	4 0907	0 9999	0 0002
5 0	4 2906	1 0000	0

Table 11 $\beta = -0\ 475$

η	f	f'	f''
0	0	0 8664	0
0 2	0 1734	0 8687	0 0233
0 4	0 3478	0 8755	0 0442
0 6	0 5239	0 8861	0 0609
0 8	0 7024	0 8995	0 0720
1 0	0 8838	0 9145	0 0769
1 2	1 0683	0 9299	0 0761
1 4	1 2557	0 9446	0 0705
1 6	1 4460	0 9579	0 0616
1 8	1 6388	0 9692	0 0510
2 0	1 8336	0 9783	0 0401
2 2	2 0300	0 9853	0 0300
2 4	2 2276	0 9904	0 0214
2 6	2 4260	0 9940	0 0146
2 8	2 6251	0 9964	0 0095
3 0	2 8245	0 9979	0 0059
3 2	3 0242	0 9988	0 0035
3 4	3 2240	0 9994	0 0020
3 6	3 4240	0 9997	0 0011
3 8	3 6239	0 9998	0 0006
4 0	3 8239	0 9999	0 0003
4 2	4 0239	1 0000	0 0001

probably the more important and one where one of the most useful tools in boundary-layer calculations, namely, the Mangler transformation, is lost Thus, the axially symmetric problem promises to be more difficult in the wake problems than it is in the flow over a body However, initial investigations indicate that the axially symmetric case can be attacked by methods analogous to those given here, i e , by similar solutions

Appendix

The problem is to find the value of the parameter β corresponding to the known solution $f'(\eta) \equiv 1$ of the differential equation

$$f''' + ff'' - \beta(f'^2 - 1) = 0 \qquad \text{(A1)}$$

with the boundary conditions $f(0) = 0, f''(0) = 0, f'(\infty) = 1$ We begin by setting $f(\eta) = 1 - \epsilon g(\eta)$ where $\epsilon = 1 - f'(0)$ and is assumed small The linearized equation for $g(\eta)$ is

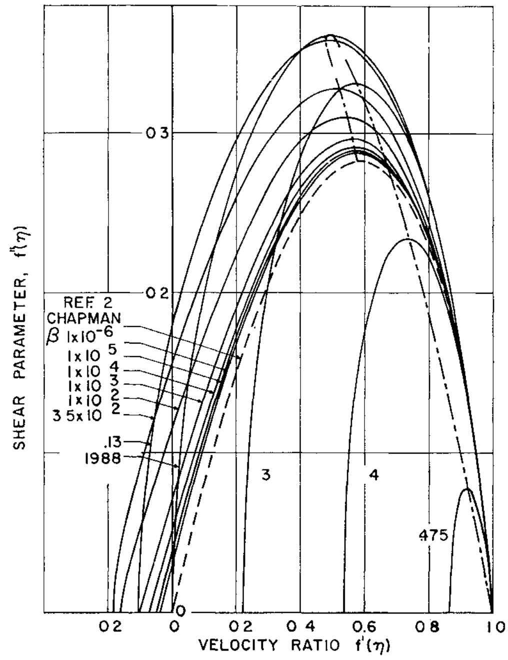

Fig 6 Shear-velocity diagram

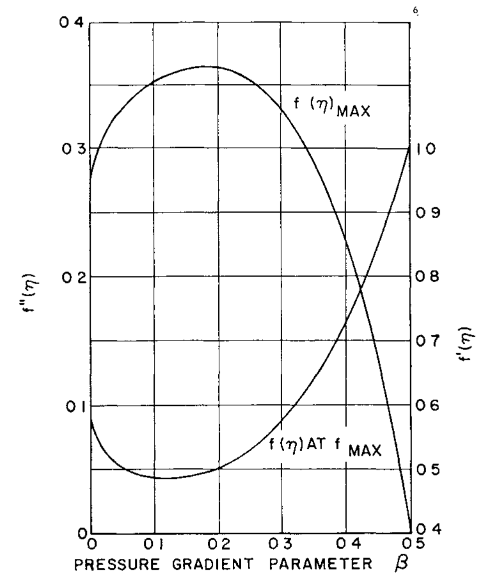

Fig 7 Maximum shear and velocity ratio at maximum shear

then

$$g'' + \eta g' - 2\beta g = 0 \qquad \text{(A2)}$$

with the boundary conditions $g'(0) = 0, g(\infty) = 0$

Setting $g(\eta) = e^{-\xi^2} H(\xi)$, $\xi = \eta/2^{1/2}$ results in Weber's equation

$$H'' - 2\xi H'' - 2(2\beta + 1)H = 0 \qquad \text{(A3)}$$

This equation has been much studied in the treatment of the quantum mechanical oscillator [9]

The substitution

$$H = \sum_0^\infty a\ \xi^\nu$$

leads to the recursion relation

$$a_{\nu+2} = 2\,\frac{[(2\beta + 1) + \nu]}{(\nu + 1)(\nu + 2)}\,a_\nu \qquad \text{(A4)}$$

which becomes $a_{\nu+2} = (2/\nu)a_\nu$ for large ν

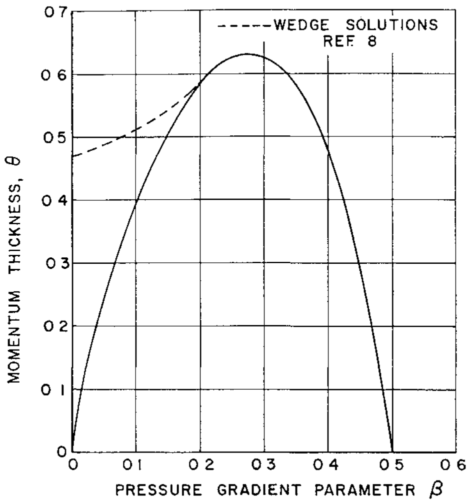

Fig 8 Momentum thickness as function of β

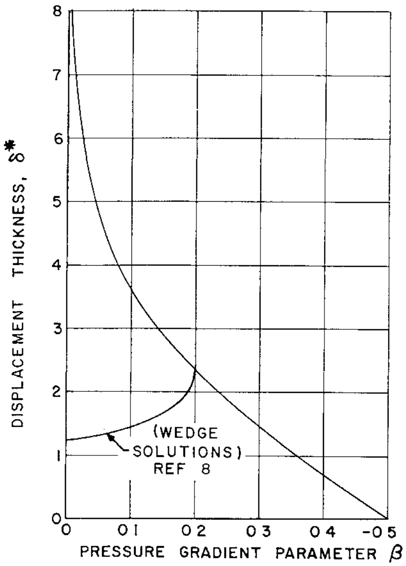

Fig 9 Displacement thickness as function of β

Now this is the relationship between coefficients in the expansion of e^{ξ^2} so that for ξ large, $H(\xi)$ is indistinguishable from e^{ξ^2} Therefore, the outer boundary condition cannot be satisfied unless the series for H terminates, since only then will the exponential $e^{-\xi^2}$ insure the decay to zero as $\xi \to \infty$ The condition for the termination of the series is simply $\nu = -(2\beta + 1)$, ν a positive integer The H (ξ) are the Hermite polynomials, the first few of which are given below:

$$H_0(\xi) = 1 \qquad H_2(\xi) = 4\xi^2 - 2$$
$$H_1(\xi) = 2\xi \qquad H_3(\xi) = 8\xi^3 - 12\xi \tag{A5}$$

Since $g(\eta) = e^{\xi^2} H_\nu(\xi)$ must be even, only the polynomials of even order are permissible Thus ν is further restricted to the even positive integers If it is further required that $f'(\eta)$ be less than unity in $-\infty \leq \eta \leq \infty$, then $g(\eta)$ must be positive in this range and any $H_\nu(\xi)$ which takes on negative values in $-\infty \leq \eta \leq \infty$ should be discarded Obviously this last restriction eliminates all but $H_0(\xi)$ The final solution is $g(\eta) = e^{-\xi^2}$ or $f'(\eta) = 1 - \epsilon e^{-\xi^2}$ and $\beta = -\frac{1}{2}$

Since $f''(\eta)$ vanishes at the origin and at infinity, there is a point of maximum shear in $0 \leq \eta \leq \infty$ Ordinarily the position of this point must come from the numerical computation However, for the linearized solution, $f''(\eta)$ reaches its maximum of $\epsilon e^{-1/2}$ at $\eta = 1$ as may be verified from the solution just calculated

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